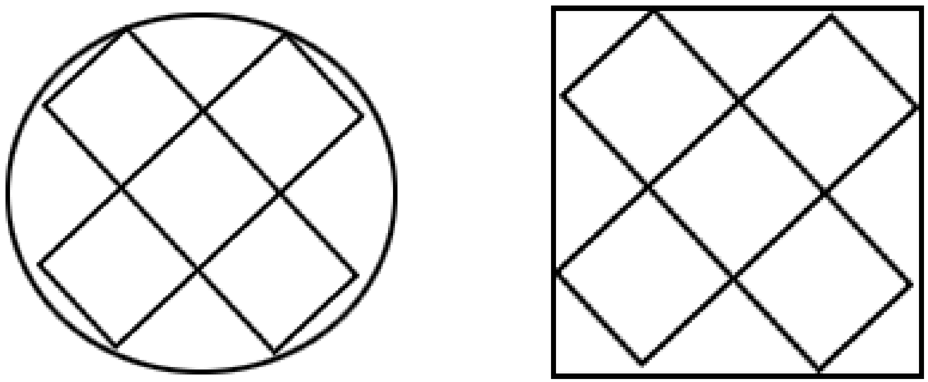
AHSMC 2011

Part II

Problem 1.
A cross shaped figure is made up of five unit squares. Determine which has the larger area: the square containing the cross or the circle containing the cross.



Problem 2.

If there is exactly one triplet (x, y, z) satisfying $x^2 + y^2 = 2z$ and $x + y + z = t$, determine t.

Problem 3.

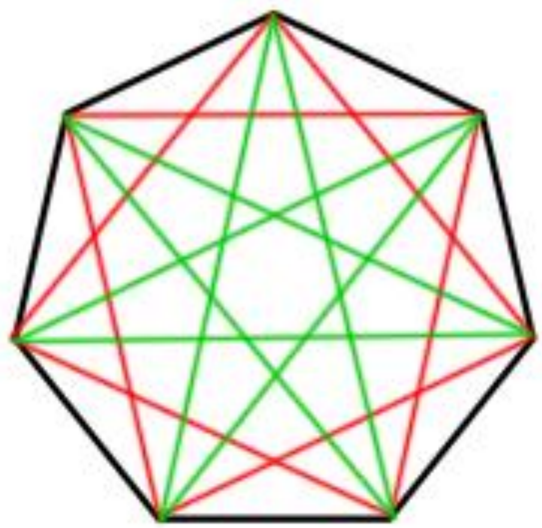
On side BC of triangle ABC, points P and Q exist such that P is closer to B than Q and $\angle PAQ = \frac{1}{2}\angle BAC$, moreover X and Y are on lines AB and AC respectively. Suppose that $\angle XPA = \angle APQ$ and $\angle YQA = \angle AQP$. Prove that $PQ = PQ + QY$.

Problem 4.

Determine all functions $f: Z \to N$ where for every n, $f(n-1)+f(n+1) \leq 2f(n)$.

Problem 5.

Seven teams gather to play one of three sports, and no set of three teams play the same sport among themselves. A triplet is considered *diverse* if all three sports are played among themselves. What is the maximum possible number of diverse triplets?



- In the above configuration, we notice there are 14 diverse triangles.
- We prove that this is actually the maximum possible configuration.
- Suppose a vertex has A black, B green, and C red edges where $A + B + C = 6$.
- Then at least $\binom{A}{2} + \binom{B}{2} + \binom{C}{2}$ non-diverse triangles exist containing this vertex.
- We are not overcounting: if the same triangle is counted twice then it is monochromatic.
- The answer is $\binom{7}{3} - 7 \times 3 = 14$.